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MEASUREMENT OF RANDOM TIME VARIANT LINEAR CHANNELS

by

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MEASUREMENT OF RANDOM TIME VARIANT LINEAR CHANNELS

ABSTRACT

This paper concerns the problems of the measureability and measurement of random time-variant linear channels. With regard to measureability, a new, less stringent channel measureability criterion is proposed to supercede the BL product introduced by Kailath. This criterion involves the area of occupancy of the Doppler-delay spread function (or its dual). By using time and bandwidth constraints on the input and output of a channel, the channel measurement problem is reduced to the measurement of a discrete set of finite parameters. Optimal measurement techniques are described and their performances determined for the two cases wherein the channel correlation function is known and unknown.

MEASUREMENT OF RANDOM TIME-VARYING LINEAR CHANNELS

I. INTRODUCTION

The measurement and characterization of communication channels has received increasing attention in recent years due to the projected widespread use of digital communications. Reliable high speed digital data transmission requires considerably more knowledge and equalization of channel characteristics than does conventional highly redundant analog transmissions. This paper is concerned both with the measureability and with the measurement of random time-varying linear channels.

The problem of the measurement of system functions (input-output relations) of random time variant channels differs from the classic problem of filtering a random signal in that even in the absence of noise the random system function may be non-measureable. It appears that this fact was first pointed out by Kailath [1] who developed some measureability criteria and introduced a channel parameter called the "spread factor". This parameter is the product of B_{max}, the maximum rate of variation of the system in Hz, by L_{max}, the maximum multipath spread of the channel. For reasons which will become clear subsequently, we call the product $B_{max} L_{max}$ the "rectangular" spread factor of the channel and denote it by S_R. Kailath demonstrated that the system functions of a linear channel can not be measured if $S_R > 1$ and if no further information than B_{max}, L_{max} are known about the channel. Unfortunately the criterion $S_R < 1$ has been uncritically accepted subsequently as the channel measureability criterion for random time-varying linear channels, without paying sufficiently careful attention to the conditions under which it was derived. In this paper we shall show that the criterion $S_R < 1$ is not the proper channel measureability criterion and we shall propose a new criterion, $S_A < 1$, where the parameter S_A

is called the area spread factor of the channel. Since it will be shown that $S_A \lessgtr S_R$, channels which were hitherto thought unmeasureable are actually measureable.

In defense of Kailath's result it is worth pointing out that our new criterion does not necessarily have to be regarded as in conflict with Kailath's results since we assume somewhat more knowledge of the fading dispersive characteristics of the channel to be measured than the gross parameters B_{max} and L_{max}. Kailath was very careful to point out that additional channel knowledge would generally allow exact channel measurement even though $B_{max} L_{max} > 1$. However, there is a moot philosophical point here: if the channel is wide sense stationary (WSS) then not only B_{max}, L_{max} but also the additional gross channel information needed in the present development may be determined without regard to the size of the product $B_{max} L_{max}$. On the other hand, if the channel is too nonstationary it is not clear that B_{max} and L_{max} can be meaningfully defined. Thus, in those cases where B_{max} and L_{max} can be meaningfully defined and measured, $B_{max} L_{max} < 1$ is too stringent a criterion to use in deciding whether the system functions of a channel may be measured.

The new channel measurement criteria and optimal measurement procedures developed here are based[*] upon the input-output representation called the <u>Doppler-Delay Spread Function</u> [2]. Discrete representations of the channel are used corresponding to simultaneous input time and output bandwidth constraints [2]. The Doppler-Delay Spread Function $V(\nu, \xi)$ provides a phenomenological model of the channel as a continuum of differential scatterers subjecting the transmitted signal to a complex gain $V(\nu, \xi) d\nu d\xi$ for scatterers providing delays and Doppler shifts

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[*] An equivalent development exists for the dual [3] system function, the <u>delay-Doppler spread function</u>, $U(\xi, \nu)$.

in the region $(\xi, \xi + d\xi) \times (\nu, \nu + d\nu)$. We define the region of the ν, ξ plane over which $V(\nu, \xi)$ is effectively nonzero as the Delay-Doppler Occupancy Pattern or simply the Occupancy Pattern channel. We have sketched four possible delay-Doppler occupancy patters in (a) - (d) of Fig. 1 with the parameters B_{max} and L_{max} explicitly indicated. The shaded area of each figure, i.e., the area of the occupancy pattern, is the new spread factor S_A. Examination of these figures show that only for the rectangular shape of Fig. 1(c) is $S_A = S_R$. The rectangular spread factor is the area of the smallest rectangle with sides parallel to the ξ, ν axes which just encloses the occupancy pattern.

Because of the discrete channel representation used, the problem of channel measurement can be reduced to estimation of a set of parameters. We consider two different approaches. In the first approach we assume the channel parameter vector to be an unknown nonrandom vector and in the second approach we assume the channel parameter vector to be a random vector selected from an ensemble with known correlation matrix. In the former case the classical least squares approach is directly applicable and has been used first by Levin [4] to estimate the impulse response of time-invariant filters. In the latter case we enter the domain of estimation theory originally made popular by Wiener [5]. For both cases we determine the structure of the optimum estimators and their performances.

The work reported here attacks a problem area similar to that studied by Bar-David [6] who considered the direct estimation of $V(\nu, \xi)$, primarily for radar targets. In the case of stationary and most physical communication channels $V(\nu, \xi)$ has the character of nonstationary white noise in the ξ and ν variables and the direct measurement of $V(\nu, \xi)$ does not appear to be a useful occupation. However, our use of input time and output bandwidth constraints leads to discrete channel models whose parameters are finite and whose measurements are meaningful.
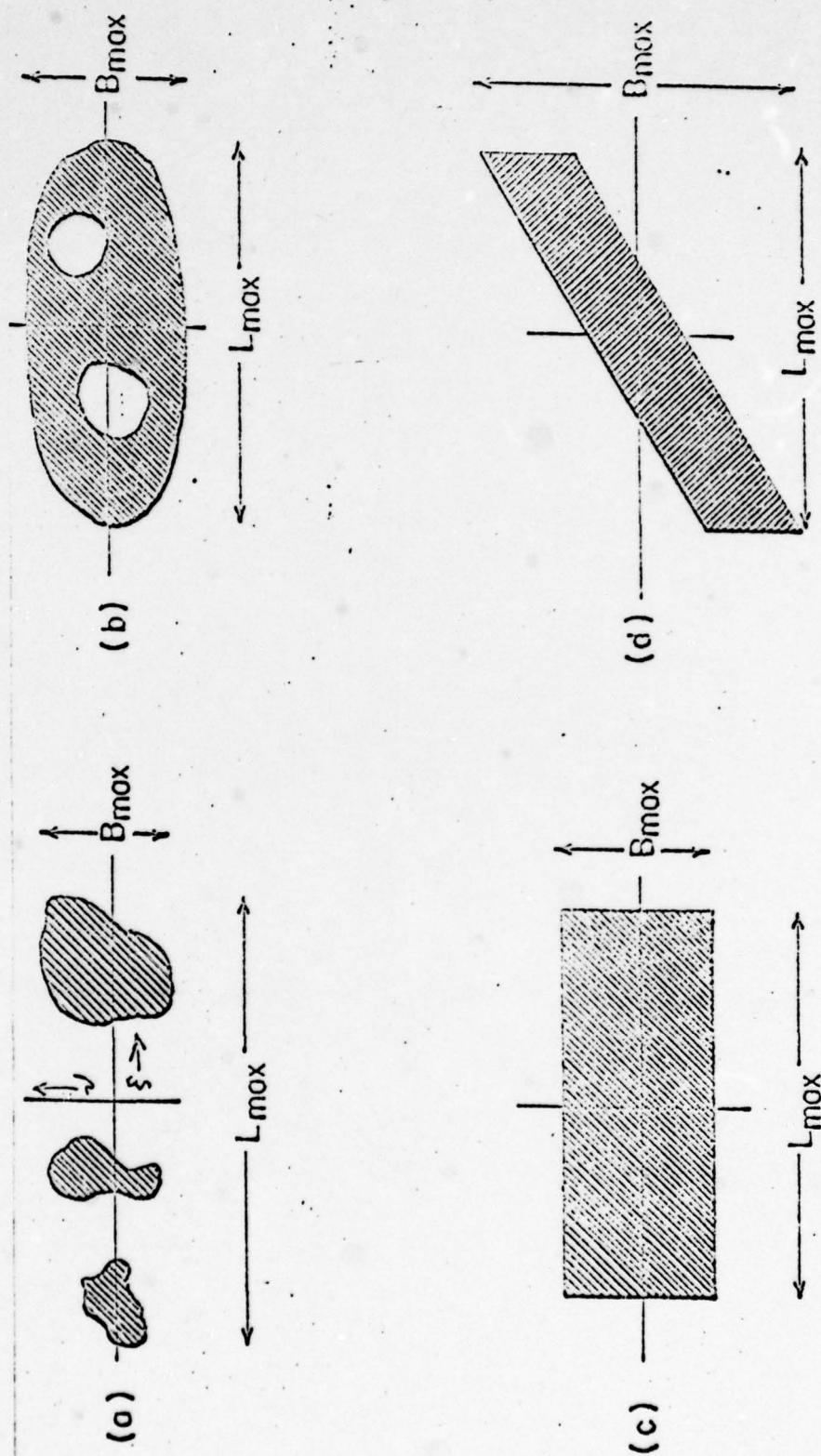
FIG. 1 EXAMPLES OF POSSIBLE DELAY DOPPLER OCCUPANCY PATTERNS.

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II. CHANNEL REPRESENTATION

There exist many different system functions for representing the input-output of time varying linear channels. Various schemes of categorization have been presented [1][2][7], but the author is partial to his own [2] and assumes that the reader has access to this work. For the development in the present paper we need only the input-output relation corresponding to the Doppler-delay spread function, i.e.,

$$w(t) = \iint z(t-\xi)e^{j2\pi\nu(t-\xi)}V(\nu,\xi)d\nu d\xi \tag{1}$$

where $z(t)$ is the input and $w(t)$ is the output[2].

When time and/or bandwidth constraints exist upon the input and/or output of a channel it is possible to replace the original channel by a simplified canonical channel model which has the same input-output behavior as the actual channel. Such channel models were first described by Kailath [1] and further developed by the author who in [2] introduced the models used in the following discussion. Following the same general procedure as used by Kailath to derive his measurability criterion we derive the new channel measurability criterion by placing certain constraints upon the input and output of the channel and employing a corresponding canonical channel model representation for the original channel. Our results will differ from Kailath's because the channel model we use, based upon $V(\nu,\xi)$ allows a simple direct inclusion of the more complicated Delay-Doppler Occupancy Patterns such as shown in Fig. 1.

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[2]In this paper we use complex envelope notation throughout without explicitly indicating so in the text Thus $z(t)$ and $w(t)$ are the complex envelopes of the actual real input and output processes.

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In [2] (Section VI A(c)) it is demonstrated that if the
input to a channel is confined by a time gate to the time inter-
val $t_i - \frac{T}{2} < t < t_i + \frac{T}{2}$ and the output spectrum is confined by
a band pass filter to the frequency interval $f_o - \frac{W}{2} < f < f_o + \frac{W}{2}$,
then in place of the actual channel with Doppler-Delay Spread
Function $V(\nu, \xi)$ one may use a channel whose Doppler-Delay Spread
Function $\tilde{V}(\nu, \xi)$ has the singular form,

$$\tilde{V}(\nu, \xi) = \sum_{m,n} V_{mn} \delta(\nu - \frac{m}{T}) \, \delta(\xi - \frac{n}{W}) \tag{2}$$

where $\delta(\cdot)$ is the unit impulse,

$$V_{mn} = \iint e^{j2\pi t_i (\nu - \frac{m}{T})} \, e^{-j2\pi f_o (\xi - \frac{n}{W})}$$

$$\text{sinc}[T(\nu - \frac{m}{T})] \, \text{sinc}[W(\xi - \frac{n}{W})] \, V(\nu, \xi) d\nu \, d\xi \tag{3}$$

and

$$\text{sinc } x = \frac{\sin \pi x}{\pi x} \tag{4}$$

An alternate expression for the gain coefficients V_{mn} is
given by

$$V_{mn} = \frac{1}{TW} \int_{t_i - \frac{T}{2}}^{t_i + \frac{T}{2}} \int_{f_o - \frac{W}{2}}^{f_o + \frac{W}{2}} e^{-j2\pi \frac{m}{T} t} \, e^{j2\pi \frac{n}{W} f} \, M(t,f) dt \, df \tag{5}$$

where $M(t,f)$ is the Frequency Dependent Modulation Function [2].

Examination of (5) reveals that V_{mn} is the Fourier coeffi-
cient in a bivariate Fourier series expansion of $M(t,f)$ in the

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time-frequency interval $(t_i - \frac{T}{2} < t < t_i + \frac{T}{2}, \; f_o - \frac{W}{2} < f < f_o + \frac{W}{2})$. Also, from (3) we see that the gain coefficient V_{mn} is essentially a two dimensional sampled version of the original Delay-Doppler Spread Function, the sampling taking place with "pulses" of width of the order of $1/T$ in the ν direction and $1/W$ in the ξ direction.

The discrete channel model, which consists of a finite set of delays and Doppler shifts, is shown in Fig. 2, where we have used the notation

$$\text{Rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ \\ 0 & |x| \geqq \frac{1}{2} \end{cases} \tag{6}$$

If we define the input to the band limiting filter as $w_1(t)$ and the gated channel input of duration T as $z_1(t)$, then (2) tells us that the following discrete channel relationship applies:

$$w_1(t) = \sum_m \sum_n z_1(t - \frac{n}{W}) \; e^{j2\pi \frac{m}{T}(t - \frac{n}{W})} \; V_{mn} \tag{7}$$

The relationship between the actual output $w(t)$ and input $z(t)$ is somewhat more involved. Form Fig. 2 we note that the actual output $w(t)$ is obtained by passing $w_1(t)$ through a bandpass filter of bandwidth W cps centered on f_o cps. To simplify the notation we shall assume $f_o = 0$, i.e., the receiver is centered on the nominal "carrier" frequency. Then

$$w(t) = w_1(t) \, \circledX \, W \, \text{sinc} \, Wt \tag{8}$$

If the bandwidth of $w_1(t)$ is less than or equal to W, then

$$w(t) = w_1(t) \tag{9}$$

and the input-output relation becomes

$$w(t) = \sum_m \sum_n \text{Rec}\left(\frac{t-\tau_i-\frac{n}{W}}{T}\right) z\left(t-\frac{n}{W}\right) e^{j2\pi\frac{m}{T}\left(t-\frac{n}{W}\right)} V_{mn} \tag{10}$$

Since the input signal is time gated and thus not bandlimited, strictly speaking $w_1(t)$ cannot be bandlimited and (10) cannot be used. However, since we assume that both the bandwidth of $z(t)$ and W greatly exceed $1/T$, the bandwidth broadening caused by the time gating has no practical significance. One may then assume that the bandwidth of $w_1(t)$, W_1, equals the sum of the bandwidth of $z(t)$, W_z, plus the maximum Doppler spreading of the channel, as determined from the Delay Doppler Occupancy Pattern. We shall generally assume

$$W_1 = W \tag{11}$$

and use the simplified input-output relationship (10)

In closing this section we should like to discuss some statistical relationships between the coefficients V_{mn} and also the way these coefficients vary with the location (t_i, f_o) of the time-frequency rectangle over which the channel is being represented. To simplify the discussion we shall assume a WSSUS (wide sense stationary uncorrelated scattering) channel for which [2]

$$\overline{V*(\nu,\xi)\ V(\mu,\eta)} = S(\xi,\nu)\ \delta(\eta-\xi)\ \delta(\mu-\nu) \tag{12}$$

where $S(\xi,\nu)$ is the Scattering Function of the channel.

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It is shown in [2] that

$$\overline{V_{mn}^* V_{rs}} \approx \begin{cases} 0 & ; \ m \neq r \ , \ n \neq s \\[2em] \frac{1}{TW} \ S(\frac{n}{W}, \frac{m}{T}) & ; \ m = r \ , \ n = s \end{cases} \qquad (13)$$

when the Scattering Function varies very little for changes in ξ of the order of $1/W$ and changes in ν of the order of $1/T$. Thus for the WSSUS channel and a sufficiently smooth Scattering Function the gains of the discrete point "scatterers" become uncorrelated and the strength of the reflection from a particular scatterer becomes proportional to the amplitude of the Scattering Function at the same value of Delay and Doppler shift.

The gain coefficient V_{mn} may be regarded as a random process which varies with t_i and f_o. For notational convenience we drop the subscripts and write

$$V_{mn}(t,f) = \iint e^{j2\pi t(\nu - m/T)} \ e^{-j2\pi f(\xi - n/W)}$$

$$\text{sinc}[T(\nu - m/T)] \ \text{sinc}[W(\xi - n/W] \ V(\nu, \xi) d\nu \xi \qquad (14)$$

The correlation function of $V_{mn}(t,f)$ is readily found to be

$$\overline{V_{mn}^*(t,f) \ V_{mn}(t+\tau, f+\Omega)} \equiv R_{mn}(\tau, \Omega)$$

$$= \iint e^{j2\pi\tau(\nu - \frac{m}{T})} \ e^{j2\pi\Omega(\xi - \frac{n}{W})}$$

$$\text{sinc}^2[T(\nu - \frac{m}{T})] \ \text{sinc}^2[W(\xi - \frac{n}{W})] \ S(\xi, \nu) \ d\xi \ d\nu \qquad (15)$$

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Assuming that T and W are large enough so that $S(\xi, \nu)$ varies little in a ξ interval $1/W$ and a ν interval $1/T$, the sinc functions in the double integral become impulsive by comparison to $S(\xi, \nu)$. Then we have the following simple result:

$$\frac{R_{mn}(\tau, \Omega)}{R_{mn}(0,0)} \approx \begin{cases} (1 - \frac{|\tau|}{T})(1 - \frac{|\Omega|}{W}) & ; \ |\tau| < T \ , \ |\Omega| < W \\ \\ 0 & ; \ |\tau| \geqq T \ , \ |\Omega| \geqq W \end{cases} \qquad (16)$$

III. NECESSITY OF AREA SPREAD FACTOR CRITERION

In this section we shall demonstrate the necessity of the area spread factor criterion. For $T \gg L_{tot}$ and $W \gg B_{tot}$ the number of V_{mn} coefficients which are significantly different from zero will be determined by how many rectangels of dimension $1/TW$ can be fitted into the Delay-Doppler Occupancy Pattern, i.e., into regions of the ν, ξ plane over which $V(\nu, \xi)$ is significantly different from zero. We have already illustrated such regions in Fig. 5 and defined the area of such a region to be S_A. In order to ignore edge effects and thereby simplify our discussion we assume $T \gg L_{tot}$, $W \gg B_{tot}$. Then the number of V_{mn} coefficients of significant amplitude can be expressed as

$$N_{ch} = S_A(TW + \alpha) \tag{17}$$

where α is a number accounting for edge effects which becomes negligible compared to TW as T and W become large.

If the multipath spread of the channel is L_{tot} then the duration of the output signal is $T + L_{tot}$ and the number of linearly independent (complex) samples is approximately $W(T + L_{tot})$ if W is taken as the bandwidth of the received process. Assuming W,T large we may express the number of independent observations as

$$N_{out} = WT + \beta \tag{18}$$

where β is a number accounting for edge effects (including samples over L_{tot}) which becomes negligible compared to TW as T,W becomes large.

Clearly, in order to be able to solve for the N_{ch} unknowns

$$N_{ch} \leq N_{out} \tag{19}$$

or

$$S_A \leq \frac{WT + \beta}{WT + \alpha} \to 1 \qquad \text{for } TW \gg 1 \qquad (20)$$

which demonstrates the necessity of the area spread factor measurability criterion.

IV. SUFFICIENCY OF AREA SPREAD FACTOR CRITERION

In order to demonstrate the sufficiency of the area spread factor criterion it is necessary to demonstrate a procedure whereby the set $\{G_{mn}\}$ may be determined if $S_A \leq 1$.

We assume that the Delay-Doppler Occupancy Pattern of the channel is bounded by a rectangle whose lower left hand coordinates are $(m_o/T, n_o/W)$. Thus,

$$B_{tot} \approx \frac{M}{T} \tag{21}$$

$$L_{tot} \approx \frac{N}{T} \tag{22}$$

In the discussion to follow we adopt the notation:

$$w_\ell = w(t_i - \frac{T}{2} + \frac{\ell}{W} + \frac{n_o}{W}) \tag{23}$$

$$z_p = z_1(t_i - \frac{T}{2} + \frac{p}{W}) \tag{24}$$

$$y_p = z_p \, e^{jm_o\theta} \tag{25}$$

$$G_{mn} = V_{m_o+m,\, n_o+n} \, e^{j(m+m_o)\theta p_i} \tag{26}$$

$$P \approx TW \tag{27}$$

$$\theta = \frac{2\pi}{TW} \tag{28}$$

$$p_i = (t_i - \frac{T}{2})W \tag{29}$$

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If the output is sampled at the Nyquist rate the resulting set of samples $\{w_\ell\}$ provides the minimum number of parameters needed to represent the received signal. In terms of the foregoing notation

$$w_\ell = \sum_{n=0}^{M} \sum_{m=0}^{M} y_{\ell-n} \, e^{jm\theta(\ell-n)} G_{mn} \tag{30}$$

Since m_0 and p_i are assumed known, knowing $\{y_p, G_{mn}\}$ is equivalent to knowing $\{z_p, V_{mn}\}$. However, we have chosen to deal with the former because of the simplified notation. If the input waveform is known (30) provides a set of $P + N + 1$ linear equations in the unknown channel gain coefficients $\{G_{mn}\}$ (or $\{V_{mn}\}$). The number of G_{mn} is approximately TWS_A.

Equation (30) may be expressed in the form

$$[w] = [Y][G] \tag{31}$$

where the matrix $[Y]$, partitioned into column vectors, has the form

$$[Y] = [Y_{00} | Y_{10} | \ldots | Y_{M0} | Y_{01} | Y_{11} | \ldots | Y_{M1} | \ldots | Y_{0N} | Y_{1N} | \ldots | Y_{MN}] \tag{32}$$

and the column vectors w, Y_{mn}, and G are given by

$$[w] = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_{P+N} \end{bmatrix} \tag{33}$$

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$$[Y_{mn}] = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ - \\ y_0 \\ y_1 \, e^{jm\theta} \\ \cdot \\ \cdot \\ \cdot \\ y_p \, e^{jm\theta_p} \\ \cdot \\ \cdot \\ \cdot \\ y_P \, e^{jm\theta P} \\ \overline{0} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

(34)

$$[G] = \begin{bmatrix} G_{00} \\ G_{10} \\ \cdot \\ \cdot \\ \cdot \\ G_{M0} \\ G_{01} \\ G_{11} \\ \cdot \\ \cdot \\ \cdot \\ G_{M1} \\ \cdot \\ \cdot \\ \cdot \\ G_{0N} \\ G_{1N} \\ \cdot \\ \cdot \\ \cdot \\ G_{MN} \end{bmatrix}$$

(35)

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If none of the $\{G_{mn}\}$ ($0 \leqq m \leqq M$, $0 \leqq n \leqq N$) are zero, then the occupancy pattern has the rectangular shape of Fig. 1(c) and the area spread factor is the same as Kailath's spread factor. When some of the $\{G_{mn}\}$ are known to be zero, the matrix $[Y]$ may be simplified by omitting columns - the column Y_{qp} being omitted if G_{qp} is zero.

In order to prove the sufficiency of the area spread factor criterion it is necessary to find at least one transmitted signal, i.e., at least one vector $\{y_0, y_1, \ldots, y_p\}$, for which the set of equation (31) is consistent. These equations will be consistent if the vectors $\{Y_{mn}\}$ constitute a linearly independent set. A necessary and sufficient condition for a set of vectors to be linearly independent is that the Gramian of these vectors does not vanish. The Gramian is the determinant of the matrix $[E]$ given by

$$[E] = [Y*]^T [Y] \tag{36}$$

The matrix $[E]$ is Hermitian symmetric. It is also non-negative definite since, if $[X]$ is a column vector

$$[X*]^T [E] [X] = [Y*X*]^T [YX] \geqq 0 \tag{37}$$

We now examine the structure of the matrix $[E]$. Consider the typical term in the matrix:

$$C_{mn,rs} = [Y_{mn}^*]^T [Y_{rs}] \tag{38}$$

From (34) we note that Y_{mn} differs from Y_{00} in a time shift of n units (n/W sec) and a frequency shift of m units (m/T cps). The inner product (38) is thus found to differ from the ambiguity function of Y_{00} only by a phase factor, i.e.,

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$$C_{mn,rs} = e^{-j(s-n)r\theta} \ \chi\left(\frac{s-n}{W}, \frac{r-m}{T}\right) \tag{39}$$

where

$$\chi\left(\frac{q}{W}, \frac{\ell}{T}\right) = \sum_{0}^{TW} y_p^* \ y_{p-q} \ e^{jp\ell\theta}$$

$$= e^{-jp_1\ell\theta} \ W \int z_1^*(t) \ z_1\left(t - \frac{q}{W}\right) e^{j2\pi\frac{\ell}{T}t} \ dt \tag{40}$$

in which the integral representation follows from an application of the sampling theorem.

One way to make the matrix [E] nonsingular is to force the off-diagonal terms to be sufficiently small compared to the diagonal terms. The diagonal terms are all equal

$$C_{mn,mn} = \Sigma|y_p|^2 = \int |z_1(t)|^2 \ dt = \int_{-T/2}^{T/2} |z(t)|^2 \ dt \tag{41}$$

If one could find a time function of effective bandwidth $W - \epsilon$ (with $\epsilon \to 0$) and time duration T whose ambiguity function could be made sufficiently small at the set of lattice points $(\frac{n}{W}, \frac{m}{T})$ (excluding (m=0,n=0) of course), then apart from a constant factor [E] would be essentially an identity matrix. Although there are undoubtedly simpler ways in which to make [E] nonsingular, there are sound reasons for constructing the [E] matrix with small off-diagonal terms. In the following section it will be shown that such a choice minimizes the variance of the measurement error in the presence of noise and greatly simplifies the structure of the optimum estimator.

It should be recognized that the ambiguity function of a signal with time duration T and bandwidth W cannot be forced to be identically zero at the complete set of lattice points since

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$$\sum_{0}^{TW} \sum_{0}^{TW} |\chi(\frac{q}{W}, \frac{\ell}{T})|^2 \approx WT |\chi(0,0)|^2 \tag{42}$$

When $\{y_p\}$ is chosen to be a maximal length shift register sequence of ± 1's it is readily shown from the ambiguity function of this sequence [9] that

$$\frac{|c_{pq,rs}|}{c_{mn,mn}} \approx \frac{1}{\sqrt{TW}} \quad : \quad p \neq r, \ q \neq s, \ TW \gg 1 \tag{43}$$

From (42) we see that this is the smallest level that the ambiguity function can have if all the side lobes are uniformly small. If only a small fraction are to be made uniformly small, as in the present case, and the others are unconstrained it should be possible to make them considerably smaller than $1/\sqrt{TW}$. Some results of Price and Hofstetter [10] are suggestive here. Their work suggests that it should be possible to clear away a symmetric convex region of low level around the origin of the ambiguity function as long as this area is less than four. Since we deal with more general areas their results would have to be generalized to be made applicable. Equation (43) states that the off-diagonal terms of [E] can be made as small as desired relative to the diagonal terms. Unfortunately, since the order of the matrix [E] is TWS_A, where S_A is the area spread factor, the reduction in size of the off-diagonal terms is accompanied by an increase in order of the matrix [E] and the well known techniques for determination of matrix singularity from inspection of the elements of a matrix, such as those dealing with diagonally dominant matrices [11], cannot be applied.

There is a simple physical argument for asserting that when $S_A > 1$ there must exist transmitted sequences $\{y_p\}$ for which [E] is invertible. [E] will be invertible if it is positive definite, i.e., if the quadratic form in (37) is positive no matter what

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vector X is chosen. But this quadratic form $[Y*X*]^T [YX]$ is just the total energy at the output of the channel when the input is $[Y]$ and the channel gains are $[X]$. Since $[Y]$ is of length TW and $[X]$ of length $S_A TW$, when $S_A < 1$ the dimensionality of $[X]$ is less than that of $[Y]$. While degenerate forms of $[Y]$ could be chosen such that $[X]$ could be adjusted to force the output to zero for all time, it is intuitively clear that there must be infinite sets of suitably chosen $[Y]$ for which no $[X]$ can force the output to zero.

Assuming that $[E]$ is invertible, we may solve for $[G]$ with the aid of $[E]^{-1}$ by noting that if (31) is pre-multiplied by $[Y*]^T$ and then by $[E]^{-1}$ there results

$$[G] = [E]^{-1} [Y*]^T [W] \qquad (44)$$

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V. MEASUREMENT IN THE PRESENCE OF NOISE: UNKNOWN CHANNEL CORRELATION FUNCTIONS

We consider now the problem of measuring the coefficients $\{G_{mn}\}$ in the presence of noise. The ℓth observed signal sample becomes

$$w_\ell = \sum_{n=0}^{N} \sum_{m=0}^{M} Y_{\ell-n} \, e^{jm\theta(\ell-n)} \, G_{mn} + n_\ell \; ; \; \ell=0,1,2,\ldots,P+N+1$$

(45)

or in vector notation

$$[w] = [Y][G] + [N]$$

(46)

where n_ℓ are the noise samples and N is the corresponding noise vector.

The method of least squares selects as estimates of $\{G_{mn}\}$ those parameters $\{\hat{G}_{mn}\}$ which make the vector

$$[w] = [Y][\hat{G}]$$

(47)

as close as possible to the observed vector in the sense of minimizing the energy of the difference vector, $\hat{w} - w$. These estimates satisfy the equations

$$[Y^*]^T \, Y[\hat{G}] = [Y^*]^T \, [w]$$

(48)

or in terms of the E matrix

$$[\hat{G}] = [E]^{-1} \, [Y^*]^T \, [w]$$

(49)

If, by proper selection of the transmitted signal [E] can be made essentially a unit matrix, i.e.,

$$[E] \approx [I] \tag{50}$$

where we have normalized

$$\sum_{p=0}^{P} |y_p|^2 = 1 \tag{51}$$

then the estimate becomes

$$[G'] = [Y^*]^T [w] \tag{52}$$

An examination of the right hand side of (52) reveals that when [E] is an identity matrix the estimator is realizable as a matched filter bank. Thus, from (52) and (36) the typical estimate G'_{mn} is given by

$$G'_{mn} = [Y^*_{mn}]^T [w] = \sum w_k y^*_{k-n} e^{-jm(k-n)\theta}$$

$$= W \int e(t) y^*(t-\frac{n}{W}) e^{-j\frac{2\pi m}{T}(t-\frac{n}{W})} dt \tag{53}$$

where, to simplify notation, we have defined

$$y(t) = z_1(t_i - \frac{T}{2} + t) e^{jm_o \theta} \tag{54}$$

In words, G_{mn} is estimated by passing the received waveform through a filter matched to a time and frequency shifted version

of $y(t)$, where the time shift is n/W seconds and the frequency shift is m/T cps. If [E] cannot be regarded as a unit matrix the above matched filter bank must be followed by the linear operation $[E]^{-1}$.

Aside from ease of instrumentation there is another reason for trying to make [E] a unit matrix: for a given transmitted signal energy and white noise the estimation error variance is minimized if [E] can be made a unit matrix. The proof is essentially identical to Levin's proof [4] for the time-invariant channel. For white noise the covariance matrix of the estimate vector $[\hat{G}]$ is given by

$$[\text{Cov } \hat{G}] = \overline{[\hat{G}\ \hat{G}^*]^T} - \overline{[\hat{G}]}\ \overline{[\hat{G}^*]^T}$$

$$= \sigma_n^2 [E]^{-1} \tag{55}$$

where

$$\sigma_n^2 = \overline{|n_\ell|^2} \tag{56}$$

Levin [4] shows that if the diagonal terms of a positive definite matrix A are unity, the diagonal elements of the inverse matrix A^{-1} will reach the minimum values of 1 if and only if A is a unit matrix. Since the diagonal terms of $[\text{Cov } \hat{G}]$ are just the error variances, these will be minimized if E can be made into a unit matrix. Thus the minimum estimation error is

$$\epsilon^2 = \Sigma\Sigma \ \overline{|G_{mn} - G_{mn}|^2} \geq \sigma_n^2 \ \text{TWS}_A \tag{57}$$

For the convenience of the engineer we now compute an input and output SNR assuming a WSSUS channel. As our output SNR we may take the ratio

$$\rho_{out} = \frac{\overline{\Sigma\Sigma|G_{mn}|^2}}{\epsilon^2} \leqq \frac{\overline{\Sigma\Sigma|G_{mn}|^2}}{TWS_A\sigma_n^2} = \frac{\Sigma\Sigma S(\frac{n}{W},\frac{m}{T})}{(TW)^2 S_A \sigma_n^2}; \quad \sum_{p=0}^{P}|y_p|^2 = 1 \tag{58}$$

where $S(\xi,\nu)$ is the Scattering Function and we have made use of (13).

As the received signal power we take the ratio of received signal energy to the time interval T. Then if the noise power is computed in the observation bandwidth, W, we find that the input SNR is given by

$$\rho_{in} = \frac{\overline{\Sigma\Sigma|G_{pc}|^2}}{TW\sigma_n^2} = \frac{\underset{m}{\Sigma}\underset{n}{\Sigma} S(\frac{q}{W},\frac{p}{T})}{(TW)^2 \sigma_n^2} \tag{59}$$

and the ratio of output to input SNR bounded by

$$\frac{\rho_{out}}{\rho_{in}} \leqq \frac{1}{S_A} \tag{60}$$

where S_A is the area spread factor, and the equality obtains when [E] is an identity matrix.

VI. MEASUREMENT IN THE PRESENCE OF NOISE: KNOWN CHANNEL CORRELATION FUNCTIONS

The previous section was concerned with optimal channel measurement for unknown channel correlation function. On the basis of experimental evidence and theoretical considerations it appears reasonable to postulate a Quasi WSSUS channel model for many time variant channels. In this section we shall assume such a channel model and white Gaussian noise and determine a minimum mean squared error linear channel estimator. When the channel fluctuations are complex Gaussian the resulting estimator is also the unconstrained minimum mean square estimator. The approach used is called Bayes estimation and may be found extensively in the literature. A recent reference is Balakrishnan [8]. To describe the optimum estimator we must define the moment matrix of the $\{G_{mn}\}$.

$$[\Lambda] = \overline{[G][G^*]^T} \tag{61}$$

For WT large we may use (13), which makes the Λ matrix diagonal with the typical term on the diagonal given by

$$\overline{|G_{mn}|^2} = \frac{S(\frac{n}{W}, \frac{m}{T})}{TW} \tag{62}$$

For white noise the optimal estimate is given by

$$[\tilde{G}] = \{[Y][\Lambda][Y^*]^T + \sigma_n^2[I]\}^{-1} [\Lambda][Y^*]^T [w] \tag{63}$$

The inversion in (63) can always be performed since the unit matrix $[I]$ is positive definite and the sum of two non-negative definite matrices is positive definite if either is positive fefinite. By factoring matrices Eq. (63) may be expressed in the alternate form

$$[\tilde{G}] = \{\sigma_n{}^2[\Lambda]^{-1} + [E]\}^{-1}[Y^*]^T[w] \tag{64}$$

A comparison of (64) with (49) reveals that the least squares estimate assuming the channel unknown differs from the minimum mean squared error estimate assuming the channel Gaussian only in a replacement of the matrix [E] in the former case, by [E] + $\sigma_n{}^2[\Lambda]^{-1}$. The matrix $\sigma_n{}^2[\Lambda]^{-1}$ is diagonal for TW sufficiently large and from (58) and (60) the typical term is proportioned to the area spread factor divided by the input SNR. At large input SNR or very low spread factor the two estimates (49) and (64) become identical, while at sufficiently low input SNR the matrix [E] can be ignored and the estimator becomes essentially the matched filter bank previously discussed. From (64) it is readily seen that when [E] may be regarded as diagonal, the receiver is a matched filter receiver at all SNR's with weightings dependent upon the strength of each path.

The estimator (63)(and (64)) minimizes the error term

$$\varepsilon^2 = \Sigma\Sigma\overline{|\tilde{G}_{mn} - G_{mn}|^2} \tag{65}$$

From the general results of [8], the error in our special case is

$$\varepsilon^2 = \sigma_n{}^2 \, Tr\{\sigma_n{}^2[\Lambda]^{-1} + [E]\}^{-1} \tag{66}$$

where Tr[A] is the trace of the matrix [A] (thr sum of the diagonal terms).

It appears intuitively obvious and it may be proven that the measurement error will decrease when statistical knowledge is made available. To illustrate this point in the simplest situation consider the case where [E] may be taken as the identity matrix.

Then

$$\epsilon^2 = \sigma_n^2 \Sigma\Sigma \frac{1}{1 + \sigma_n^2/|G_{mn}|^2} \tag{67}$$

which by comparison with (57) is seen to be less than the error when the channel scattering function is unknown.

The estimator discussed in this section differs from that of the previous section in that it distorts, i.e., the "signal" component of the estimator output differs from the channel complex gain vector by a linear transformation. As a consequence there are actually two types of errors: a linear distortion error and an additive noise error. The mean squared error ϵ^2 in (66) is influenced by both types of errors. From (64) and (46) it is readily seen that [H], the signal component of the estimator output is given by

$$[H] = \mathcal{E}\,[\tilde{G}] = \left[\sigma_n^2 \Lambda^{-1} + [E]\right]^{-1}[E][G] \tag{68}$$

where $\mathcal{E}$ denotes an average over the additive noise only. It is clear that the additive noise component of the estimator output, M, is given by

$$[M] = \{\sigma_n^2[\Lambda]^{-1} + [E]\}^{-1}[Y^*]^T[N] \tag{69}$$

An output signal-to-noise ratio based upon the ratio of the strength of [H] to the strength of [M] can lead to physically meaningless results because [H] and [G] are related by

$$H_{mn} = \alpha_{mn}G_{mn} + \underset{rs}{\Sigma\Sigma'}\,\beta_{mnrs}G_{rs} \tag{70}$$

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where α_{mn} and ρ_{mnrs} are constants and the prime on $\Sigma\Sigma'_{rs}$ denotes exclusion of the mn term. Ideally α_{mn} would equal 1 and the $\rho_{mnrs} \equiv 0$. In practice, however, the amplitude factor $\alpha_{mn} < 1$ and the interference term $\Sigma\Sigma'_{rs} \rho_{mnrs} G_{rs}$ will be present. When the gain coefficients G_{mn} are uncorrelated the second term in (71) may safely be regarded as "noise" and the output signal strength identified as

$$\Sigma\Sigma |\alpha_{mn}|^2 \overline{|G_{mn}|^2} \tag{71}$$

When the double sum in (70) is correlated with the first term, the signal term must be identified as the sum of the first term and that portion of the double sum correlated to the first term.

When E is the identity matrix and the G_{mn}'s are uncorrelated

$$[H] = \left[\sigma_n^2 [\Lambda]^{-1} + [I]\right]^{-1} [G] \tag{72}$$

where the matrix $\sigma_n^2 [\Lambda]^{-1} + I$ is diagonal. In particular

$$H_{mn} = \frac{1}{1 + \sigma_n^2/|G_{mn}|^2} G_{mn} \tag{73}$$

so that in this simple case there is no uncorrelated part and [H] is "all" signal. The signal output and noise powers are then

$$\overline{[H^*]^T [H]} = \Sigma\Sigma \frac{\overline{|G_{mn}|^2}}{[1 + \sigma_n^2/|G_{mn}|^2]^2} \tag{74}$$

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$$\overline{[M^*]^T[M]} = \text{Tr}\left[\overline{[M][M^*]^T}\right]$$

$$= \text{Tr}\left[\sigma_n^2 [\Lambda]^{-1} + [I]^{-2}\sigma_n^2\right]$$

$$= \Sigma\Sigma \frac{\sigma_n^2}{\left[1 + \sigma_n^2 / \overline{|G_{mn}|^2}\right]^2} \qquad (75)$$

When all gain coefficients have the same strength

$$\overline{|G_{mn}|^2} = \overline{|G|^2} \qquad (76)$$

it is readily found that

$$\rho_{out} = \frac{\overline{|G|^2}}{\sigma_n^2} = \frac{\rho_{in}}{S_A} \qquad (77)$$

which is the same as for the case wherein the Scattering Function is unknown and $[E] = [I]$ (c.f. 60).

Although in some cases knowledge of the scattering function may provide little benefit when $[E]$ is "close" to an identity matrix, one may show that when E is nearly singular the measurement error for unknown channel statistics becomes very large compared to that when the statistics are known. In particular, suppose that E is singular so that the measurement technique of the previous section fails. We now determine the performance of the present technique. To simplify the computation we shall confine our attention to the computation of the mean squared error ϵ^2 at large input SNR's.

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Let us suppose that all paths are of equal strength, so that

$$[\Lambda] = \overline{|G|^2}[I] \tag{78}$$

The matrix E being a non-negative definite Hermitian Symmetric matrix may be factored in the form

$$E = [Q^*]^T[\lambda][Q] \tag{79}$$

where Q is a unitary matrix

$$[Q^*]^T = [Q]^{-1} \tag{80}$$

and $[\lambda]$ is a non-negative diagonal matrix whose entries are the eigenvalues of [E].

Using (78) - (80) in (66)

$$\epsilon^2 = \text{Tr}\{[Q]^{-1}\left[\frac{1}{\overline{|G|^2}}[I] + \frac{1}{\sigma_n^2}[\lambda]\right]Q\}^{-1} \tag{81·}$$

Since the operation $[Q]^{-1}[\cdot][Q]$ is a similarity transformation it does not change the trace of the matrix. Thus

$$\epsilon^2 = \sum_{k=1}^{TWS_A} \frac{\overline{|G|^2}}{1 + \frac{\overline{|G|^2}}{\sigma_n^2}\lambda_k} = \sum_{k=1}^{TWS_A} \frac{\overline{|G|^2}}{1 + \frac{\rho_{in}}{S_A}\lambda_k} \tag{82}$$

where λ_k is the kth eigenvalue of E. As ρ_{in}/S_A becomes large

$$\lim \epsilon^2 \to \overline{|G|^2} \cdot 0 \quad \text{as} \quad \frac{\rho_{in}}{S_A} \to \infty \tag{83}$$

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where O is the number of zero eigenvalues. Since the percentage measurement error is given by

$$\eta = \frac{\epsilon^2}{\Sigma\Sigma|G_{mn}|^2} = \frac{\epsilon^2}{TWS_A |G|^2} \tag{84}$$

we note that for E singular and uncorrelated G'_{mn}s

$$\eta = \frac{O}{TWS_A} \tag{85}$$

which will be small if the number of zero eigenvalue is small compared to the length of the vector [G].